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The Fast-Multipole Method Applied to Inductive-Voltage Calculations in a Network Model for Rutherford-type Cables

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Abstract:
A pedestrian description is given for the Fast-Multipole Method and its application to computing the inductive voltages in an electrical-network model for Rutherford-type cables. Some numerical results are presented which demonstrate the efficiency and accuracy of the method.

Introduction:

There is presently considerable interest in improving our understanding of the stability properties of Rutherford-type cables made from Nb_3Sn strands. Recent measurements performed at Fermilab [1, 2] have shown that short cable samples, measured under self-field conditions, can have critical current degradations of 80% or more compared to expectations from measurements on individual strands. Since Nb_3Sn magnet technology will play a vital role in the LHC and future accelerators, it is imperative that the stability issues related to strands, cables, and magnets be thoroughly investigated.

Previous efforts to model quench dynamics in cables have either restricted themselves to treating a small number of strands, not modeling the true transposition pattern, or neglected the majority of inductive linkages between various parts of the cable [3-9]. The reasons for this neglect are undoubtedly due to the computational difficulty in treating the complete set of inductive interactions in a long cable with many strands. Although transposing the strands does reduce inductive couplings with respect to a changing applied field, it is well known that transposition does virtually nothing to reduce the couplings in a changing self-field [10]. It is also displeasing, from a computational perspective, to completely neglect interactions simply for lack of a better alternative.

This note describes a computational method for efficiently solving the complete system of electromagnetic equations (including the full set of inductive interactions) describing a network representation of a cable.

Problem Description:

The most detailed description of current and heat redistribution between strands in a cable can be obtained by modeling the cable as an electrical network [3, 4, 7, 9, 11]. In such a model the strands are represented as wires, and the inter-strand contacts as ohmic resistors.

Consider a cable consisting of N_s strands and having a transposition pitch length L_p. For computational purposes, each pitch length can be divided longitudinally into a set of N_s unit cells. Each unit cell contains $2N_s - 2$ sections of strand, $2N_s$ adjacent contacts, and $N_s - 1$ crossover contacts. Each contact is represented as a linear resistance, and each strand section as a resistance (either linear or nonlinear) in series with a self inductance; in addition, each pair of strand sections has a mutual inductance linking them. The complete set of electromagnetic equations, for determining the strand and contact currents, can be derived from Kirchoff's laws.

The fact that the inductance matrix is dense poses a major difficulty in the solution of the electromagnetic equations: the storage requirements for the inductance matrix scale like N^2, and a direct inversion of the matrix would require N^3 operations, where $N = N_p \cdot N_s \cdot (2N_s - 2)$ and N_p is the total number of pitches in the cable. As N_p and N_s increase, a direct inversion of the inductance matrix quickly becomes prohibitively expensive.

The only practical method for solving the complete system of equations, describing the electromagnetics of a long cable, is to use an iterative method such as conjugate gradients, or one of the other subspace projection methods [12]. In the conjugate gradients method the solution is generated as a linear combination of orthogonal vectors which span the solution space. The orthogonal vectors are generated one at a time by matrix vector products [12, 13]. At each iteration, the inductance matrix will need to be multiplied by a vector, and this matrix-vector multiplication will require N^2 operations and the storage of N^2 values. For large N the matrix-vector multiplications become a bottleneck in the computation, and we would like to reduce the operation count and storage requirements to something proportional to N instead of N^2. Both of these criteria can be realized by implementing a modern computational method called the Fast-Multipole Method (FMM) [14-17]. Another method, the Pre-corrected Fast Fourier Transform Method (PCFFTM) [17] can also be used, but it has some disadvantages with respect to FMM, such as requiring the creation of fictitious source currents by projecting the true currents onto a regular lattice.

The Fast-Multipole Method:

When the currents in the various strand sections vary in time, an electric field is induced throughout the entire cable. The inductive voltage drop, along a given strand section, is found by integrating the induced electric field along the strand section. If the vector defining strand section j is denoted by $\vec{\ell}_j$, and its current is denoted by I_j, then the induced voltage drop along strand j is given by:

$$
\begin{aligned}
V_j &= -\int_{\{j\}} \vec{E}_{ind} \cdot d\vec{\ell}_j = \int_{\{j\}} \left(\sum_{k=1}^{N} \frac{\mu_0}{4\pi} \dot{I}_k \int_{\{k\}} \frac{d\vec{\ell}_k}{r_{j,k}} \right) \cdot d\vec{\ell}_j = \sum_{k=1}^{N} M_{j,k} \dot{I}_k \\
M_{j,k} &\equiv \frac{\mu_0}{4\pi} \iint_{\{j,k\}} \frac{d\vec{\ell}_j \cdot d\vec{\ell}_k}{r_{j,k}}
\end{aligned}
\tag{1}
$$

where $r_{j,k}$ is the separation distance between the integration points on strand sections j and k. The values of $M_{j,k}$ for arbitrary orientation and placement of strand sections j and k can be found in the literature [18]. A direct evaluation of the summation in (1) will take N operations; the FMM allows this summation to be approximated, to high accuracy, with an operation count of order unity, and without having to store the majority of the $M_{j,k}$ values. Since the summation needs to be evaluated N times in a single matrix-vector multiply, the FMM reduces the storage requirement and operation count by an order of magnitude.

The basic idea of the FMM is to use a truncated multipole expansion, derived from the well-known multipole expansion of the kernal $1/r_{j,k}$ [15, 16, 19], to represent the electric field induced by a collection of strand sections; this multipole expansion can then be used to evaluate the line integrals along distant strand sections. When strand sections are too close to one another a truncated multipole expansion will become a less accurate representation of the induced electric field, therefore, for strand sections in the same local neighborhood the FMM will evaluate the pairwise interactions exactly, using the known expressions [18], and the multipole expansion is only used to compute interactions between well-separated strand sections.

The reason that the FMM is able to reduce the complexity of the matrix-vector multiply so dramatically is because it represents the effects of a large number of current sources by a short series; if the number of terms in the series is much smaller than the number of sources being represented, then the number of operations and the storage requirements become much smaller. The massive 'information compression' achieved by the FMM is possible because the multipole expansion converges so rapidly, and because more distant source currents can be resolved in progressively larger groups and represented, to the same accuracy, by a series of the same length.

Numerical Results:

To demonstrate the efficiency of the FMM, compared to a direct evaluation of the matrix-vector product, a series of tests were performed on various lengths of a 40-strand Rutherford cable. The results of the tests are shown in Table 1.

L_{cab} (m)	RMS Error (%)	CPU Time (sec)		Time Ratio
0.704	7.552×10^{-3}	FMM Direct	3.004 18.807	6.3
1.408	7.972×10^{-3}	FMM Direct	6.059 74.988	12.4
2.816	8.194×10^{-3}	FMM Direct	12.208 302.355	24.8
5.632	8.320×10^{-3}	FMM Direct	24.255 1201.287	49.5
11.264	8.404×10^{-3}	FMM Direct	48.670 4774.285	98.1

Table 1 Comparison of computation times for a single evaluation of the inductive voltage vector in a 40-strand Rutherford cable of various lengths. It can be seen that the FMM scales linearly with cable length while the direct computation scales quadratically; in addition, the FMM maintains very high accuracy (RMS Error < 0.01%).

It is clear from Table 1 that the FMM has a profound impact on the time taken to solve the linear-equation system by iteration: for an 11-meter long 40-strand cable the computation time is reduced by a factor of 100, and for longer cables the speed up will be even more dramatic.

Conclusion:

The FMM has been implemented for evaluating inductive voltages in the network model of a Rutherford-type cable, and its dramatic impact on computation times has been demonstrated. When combined with a thermal-analysis solver, this FMM-based network model will make possible the efficient modeling of stability and quench dynamics in long cables containing many strands. The resulting combined thermal-electromagnetic solver should contribute vastly to our understanding of quench propagation and traveling normal zones in cables, and the dependence of these phenomena on the distribution of material properties, strand characteristics, and transport current levels.

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